

**MATHEMATICS SOLUTION
(CBCGS SEM – 4 MAY 2018)
BRANCH – EXTC ENGINEERING**

1 a) Find the extremal of $\int_0^1 (xy + y^2 - 2y^2 y') dx$. (05)

Ans: Let $\int_{x_1}^{x_2} F dx = \int_0^1 \int_0^1 (xy + y^2 - 2y^2 y') dx$.

$$\therefore F = xy + y^2 - 2y^2 y'$$

$$\therefore \frac{\partial F}{\partial y} = x + 2y - 4yy'; \quad \& \quad \frac{\partial F}{\partial y'} = 2y^2;$$

By Euler's Lagrange equation, the necessary condition for the given functional to be extremum is $\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0$

$$\therefore x + 2y - 4yy' - \frac{d}{dx}(-2y^2) = 0$$

$$\therefore x + 2y - 4yy' + 2(2y \cdot y') = 0$$

$$\therefore x + 2y - 4yy' + 4yy' = 0$$

$$\therefore x + 2y = 0$$

$$\therefore \text{The extremal of the given function is } y = \frac{-x}{2}$$

1 b) State Cauchy – Schwartz inequality in R^3 and verify it for $u = (-4, 2, 1)$ and $V = (8, -4, -2)$ (04)

Ans: **Part.I**

Cauchy Schwartz inequality:

Statement: "If 'u' and 'v' are vectors in a real inner product space then

$$|\vec{u} \cdot \vec{v}| \leq |\vec{u}| \cdot |\vec{v}|.$$

Proof.

If \vec{u} and \vec{v} are any two vectors in R^2 , then by definition of Dot product, $\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos \theta$

$$\therefore |\vec{u} \cdot \vec{v}| = |\vec{u} \cdot \vec{v} \cos \theta| = |\vec{u}| \cdot |\vec{v}| \cdot |\cos \theta|$$

But, $-1 \leq \cos \theta \leq 1$

i.e. $|\cos \theta| \leq 1$

$$\therefore |\vec{u} \cdot \vec{v}| \leq |\vec{u}| \cdot |\vec{v}|$$

Hence, Proved.

Part.II

Let $u = (-4, 2, 1), v = (8, -4, -2)$.

$$\therefore \|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

$$= \sqrt{(-4)^2 + 2^2 + 1^2}$$

$$= \sqrt{21} \text{ and,}$$

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$= \sqrt{8^2 + (-4)^2 + (-2)^2}$$

$$= \sqrt{84}$$

$$\|\vec{u}\| \|\vec{v}\| = \sqrt{21} \times \sqrt{84}$$

$$\therefore \|\vec{u}\| \|\vec{v}\| = 42 \rightarrow (1)$$

$$\text{Also, } \langle u, v \rangle = (-4)(8) + (2)(-4) + (1)(-2)$$

$$= -32 - 8 - 2$$

$$= -42$$

$$\therefore |\langle \vec{u}, \vec{v} \rangle| = 42 \rightarrow (2)$$

From (1) & (2), we observe, $|\langle \vec{u}, \vec{v} \rangle| = \|\vec{u}\| \|\vec{v}\|$

\therefore Cauchy -Schwartz inequality is verified.

1 c) If $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are eigen values of A, then show that $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} \dots, \frac{1}{\lambda_n}$ are the eigen values of A^{-1} (05)

Ans: Given, A is non singular matrix.

$\therefore A^{-1}$ exists.

Given, λ is eigen value and X is corresponding eigen vector of matrix A.

$$\therefore AX = \lambda X$$

Pre-multiplying by A^{-1} , $A^{-1}(AX) = A^{-1}(\lambda X)$

$$\therefore (A^{-1}A)X = \lambda A^{-1}X$$

$$\therefore IX = \lambda A^{-1}X$$

$$\therefore \frac{1}{\lambda} X = A^{-1}X \text{ (Since, } IX = X \text{)}$$

$$\therefore A^{-1}X = \frac{1}{\lambda} X$$

$\therefore \frac{1}{\lambda}$ is eigen value and X is corresponding eigen vector of A^{-1}

1 d) A random variable X has the following probability mass distribution . (05)

Find c and

determine $P(X < 1)$.

X	0	1	2
P(X=z)	$3c^3$	$4c - 10c^2$	$5c-1$

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Ans:

For any probability mass function $\sum_{i=-\infty}^{\infty} P_i = 1$

$$\therefore P(0) + P(1) + P(2) = 1$$

$$\therefore 3k^3 + (4k - 10k^2) + (5k - 1) = 1$$

$$\therefore 3k^3 + 4k - 10k^2 + 5k - 2 = 0$$

$$\therefore 3k^3 - 10k^2 + 9k - 2 = 0$$

On solving we get, $k = 1, 2, \frac{1}{3}$

$$\text{But, } 0 \leq P_1 \leq 1$$

When $k = 1$, $P(0) = 3(1)^3 = 3$, which is not possible.

When $k = 2$, $P(0) = 3(2)^3 = 24$, which is not possible

$$\therefore k = \frac{1}{3}$$

$$\therefore P(0) = 3 \times \frac{1}{3^3} = \frac{1}{9}$$

$$\therefore P(1) = 4k - 10k^2 = 4\left(\frac{1}{3}\right) - 10\left(\frac{1}{3}\right)^2 = \frac{2}{9}$$

$$\therefore P(2) = 5k - 1 = 5\left(\frac{1}{3}\right) - 1 = \frac{2}{3}$$

The p.m.f. is

X	0	1	2
P(X=x)	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{3}$

$$\therefore P(x < 1) = P(0) = \frac{1}{9}$$

$$\therefore P(0 < X \leq 2) = P(1) + P(2) = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

Hence,

$$k = \frac{1}{3}; P(x < 1) = \frac{1}{9} \text{ and } P(0 < x \leq 2) = \frac{1}{3}$$

2 a) Evaluate $\int_0^{1+i} dz$ along

(06)

- i) the line $y = x$,
- ii) the parabola $x = y^2$.

Is the line integral independent of the path? Explain.

Ans: Let $f(z) = z^2$

$$\therefore u + iv = (x + iy)^2$$

$$\therefore u + iv = x^2 + i2xy + i^2y^2$$

$$\therefore u + iv = x^2 - y^2 + i2xy$$

Comparing both sides, $u = x^2 - y^2$ and $v = 2xy$

$$\therefore u_x = 2x \text{ and } v_x = 2y$$

$$\therefore u_y = -2y \text{ and } v_y = 2x$$

We observe,

1) u_x, u_y, v_x, v_y are continuous functions of x & y .

2) $u_x = v_y$ and

3) $u_y = -v_x$

\therefore Cauchy Reimann's equation are satisfied .

$\therefore f(z) = z^2$ is analytic.

\therefore Line integral is independent of the path.

$$\therefore \text{Line integral } \int_0^{1+i} z^2 dz = \left[\frac{z^3}{3} \right]_0^{1+i}$$

$$= \frac{1}{3} [(1+i)^3 - 0]$$

$$= \frac{1}{3} (-2 + 2i)$$

$$= \frac{2}{3} (-1+i)$$

∴ Line integral along the line $y = x$ or parabola $x = y^2$ or parabola $x^2 = y$ is $\frac{2}{3}(-1 + i)$.

2 b) A random variable X has the following density function $f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \leq 0. \end{cases}$ **(06)**

Find the m.g.f and hence, its mean and variance.

Ans: By definition M.G.F about origin is given by

$$M_o(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_{-\infty}^0 e^{tx} \cdot 0 dx + \int_0^{\infty} e^{tx} \cdot 2e^{-2x} dx$$

$$= 0 + 2 \int_0^{\infty} e^{-2x+tx} dx$$

$$= 2 \int_0^{\infty} e^{-(2-t)x} dx$$

$$= 2 \left[\frac{e^{-(2-t)x}}{-(2-t)} \right]_0^{\infty}$$

$$= 2 \left[0 - \frac{1}{-(2-t)} \right]$$

$$= M_o(t) = \frac{2}{2-t}$$

Expanding $M_o(t)$ as an infinite series.

$$M_o(t) = \frac{2}{2(1-t/2)}$$

$$= \left(1 - \frac{t}{2}\right)^{-1}$$

$$= 1 + \frac{t}{2} + \frac{t^2}{2^2} + \frac{t^3}{2^3} + \dots$$

$$= 1 + \frac{1!}{2} \times \frac{t}{1!} + \frac{2!}{2^2} \times \frac{t^2}{2!} + \frac{3!}{2^3} \times \frac{t^3}{3!} + \dots$$

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Now, r^{th} Moments = coefficient of $\frac{t^r}{r!}$ In M (t).

∴ First Moment about origin (μ_1')

= Coefficient of $\frac{t}{1!}$

$$= \frac{1!}{2} = \frac{1}{2}$$

∴ Second Moment about origin (μ_2')

= Coefficient of $\frac{t^2}{2!}$

$$= \frac{2!}{2^2} = \frac{1}{2}$$

∴ Mean Mean = $\mu_1' = \frac{1}{2} = 0.5$ and,

$$\text{Variance} = \mu_2' - \mu_1'^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25$$

Hence,

$$\text{M.G.F. is } M_o(t) = \frac{2}{2-t}$$

Mean = 0.5

Variance = 0.25

2 c) Calculate R (Spearman's rank correlation) and r-(Karl-Pearson's) from the following data:

x	12	17	22	27	32
y	113	119	117	115	121

Interpret your result.

(06)

Ans :

x	y	R_1	R_2	$d_i = R_1 - R_2$	d_i^2
12	113	5	5	0	0
17	119	4	2	2	4

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22	117	3	3	0	0
27	115	2	4	-2	4
32	121	1	1	0	0
				Total	8

Here, $n=5$

$$\therefore \text{Spearman's rank correlation Co-efficient } R = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \times 8}{5 \times (5^2-1)}$$

$$= 1 - 0.4$$

$$= 0.6$$

\therefore Spearman's rank correlation Co-efficient = 0.6

3 a) Let $V = R^3$. Show that W is a subspace of R^3 , where $W = \{(a,b,c) | a+b+c=0\}$ that is W consists of all vectors where the sum of their components is zero. (06)

Ans: Let \vec{u} and \vec{v} be any two vectors in W and let 'k' be any scalar.

If W is non-empty subset of V then W is sub-space if

i) $\vec{u} + \vec{v}$ is in W

ii) $k \vec{u}$ is in W .

Let $\vec{u} = (a_1, b_1, c_1)$ and $\vec{v} = (a_2, b_2, c_2)$ be any two vectors belonging to set

W such that $a_1 + b_1 + c_1 = 0 \rightarrow (1)$ & $a_2 + b_2 + c_2 = 0 \rightarrow (2)$

(a) $\vec{u} + \vec{v} = (a_1, b_1, c_1) + (a_2, b_2, c_2)$

$$\therefore \vec{u} + \vec{v} = (a_1 + a_2 + b_1 + b_2 + c_1 + c_2)$$

Consider, $(a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) = (a_1 + b_1 + c_1) = (a_2 + b_2 + c_2)$

$\therefore (a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) = 0$ (From 1 & 2)

$\therefore \vec{u} + \vec{v}$ is in **W**

(b) $k\vec{u} = k(a_1, b_1, c_1)$

$\therefore k\vec{u} = (ka_1, kb_1, kc_1)$

Consider, $ka_1 + kb_1 + kc_1 = k(a_1 + b_1 + c_1)$

$\therefore ka_1 + kb_1 + kc_1 = 0$ (From 1)

$\therefore k\vec{u}$ is in **W**

Hence, set **W** is a subspace of R^3

3 b) Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z-1|=3$. (06)

Ans: Let $f(z) = \frac{e^{2z}}{(z+1)^4}$

Here $z_0 = -1$ is a pole of order 4.

$|z|=2$ is a circle with Centre is (0,0) and radius 2

$z_0 = -1$ Lies inside the circle.

R= Residue of $f(z)$ at " $z = -1$ "

$$= \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n \times f(z)$$

$$= \frac{1}{(4-1)!} \lim_{z \rightarrow -1} \frac{d^3}{dz^3} (z+1)^4 \times \frac{e^{2z}}{(z+1)^4}$$

$$= \frac{1}{3!} \lim_{z \rightarrow -1} \frac{d^2}{dz^2} e^{2z} \cdot 2$$

$$= \frac{1}{6} \lim_{z \rightarrow -1} \frac{d}{dz} 2 e^{2z} \cdot 2$$

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$$= \frac{1}{6} \lim_{z \rightarrow -1} 4e^{2z} \cdot 2$$

$$= \frac{1}{6} \times 8e^{2(-1)}$$

$$= \frac{4}{3e^2}$$

By Cauchy's Residue theorem, $\int_c f(z) dz = 2\pi i(R_1 + R_2 + \dots)$

$$\therefore \int_c \frac{e^{2z}}{(z+1)^4} dz = 2\pi i \cdot \frac{4}{3e^2}$$

Hence, $\int_c \frac{e^{2z}}{(z+1)^4} dz = \frac{8\pi i}{3e^2}$

3 c) Show that the matrix A is diagonalizable.

(08)

Also find the transforming matrix and the diagonal matrix where $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$

Ans: Let λ be eigen value and X be corresponding eigen vector of matrix A.

Characteristic equation is $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

On solving we get,

$$\lambda^3 - (\text{sum of diagonal elements}) \lambda^2 + (\text{sum of the minors of diagonal elements}) \lambda - |A| = 0$$

$$\therefore \lambda^3 - (4 + 5 + 2)\lambda^2 + (12 + 9 + 18)\lambda - 45 = 0$$

$$\therefore \lambda^3 - 11\lambda^2 + 39\lambda - 45 = 0$$

\therefore Eigen values (λ) are 3, 3, 5

Case 1: $\lambda = 3$

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$$\therefore [A - \lambda I] X = 0$$

$$\therefore \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 - 2R_1; \quad R_3 - R_1 \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

Number of unknowns (n)=3

Rank (r) = number of non-zero rows=1

Algebraic multiplicity (A.M)

= No .of times " $\lambda = 3$ " is repeated =2

Geometric multiplicity (G.M)= n -r = 3 -1= 2

\therefore A.M =G.M for " $\lambda = 3$ "

Expanding (1), $1x_1 + 1x_2 - 1x_3=0$

Put $x_1 = t$ and $x_2 = s$

$$\therefore t + s - x_3 = 0$$

$$\therefore x_3 = t + s$$

\therefore Eigen vector X is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ s \\ t + s \end{bmatrix} = \begin{bmatrix} 1t + 0s \\ 0t + 1s \\ 1t + 1s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} s$$

$$\therefore \text{Eigen vector } X_1 = [1 \ 0 \ 1]' \text{ \& } X_2 = [0 \ 1 \ 1]'$$

Case 2: $\lambda = 5$

$$\therefore [A - \lambda I] X = 0$$

$$\therefore \begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 + 2R_1; R_3 + R_1 \Rightarrow \begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & -4 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 - R_2; \frac{1}{2}R_2 \Rightarrow \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (2)$$

Here, $n=3$ and $r = 2$

A.M. = No. of times " $\lambda = 5$ " is repeated = 1

$$G.M. = n - r = 3 - 2 = 1$$

\therefore A.M = G.M for " $\lambda = 5$ "

Expanding (2),

$$-x_1 + x_2 + x_3 = 0 \rightarrow (3) \text{ \& } x_2 - 2x_3 = 0 \rightarrow (4)$$

From (4), $x_2 = 2x_3$

From (3), $-x_1 + x_2 - x_3 = 0$

$$\therefore -x_1 + 2x_3 - x_3 = 0$$

$$\therefore x_3 = x_1$$

$$\therefore \text{Eigen vector } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} x_3$$

∴ Eigen vector $X_3 = [1 \ 2 \ 1]'$

Since, A.M = G.M for all eigen values, matrix A is diagonalizable.

$$\therefore M^{-1}AM = D$$

So, the given matrix A is diagonalizable to Diagonal Matrix D by the Transforming Matrix M, where

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ and } M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

4 a) Find the extremals $\int_{x_0}^{x_1} (2xy - y''') dx$. (06)

Ans: Let $\int_{x_1}^{x_2} f dx = \int_{x_0}^{x_1} (2xy - y''') dx$

$$\therefore f = 2xy - y'''$$

$$\therefore \frac{\partial f}{\partial y} = 2x; \quad \frac{\partial f}{\partial y'} = 0, \quad \frac{\partial f}{\partial y''} = 0; \quad \frac{\partial f}{\partial y'''} = 2y''' = -2 \frac{d^3 y}{dx^3};$$

By Euler Lagrangian corollary, the necessary condition for the given functional to be extremum is

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) - \frac{d^3}{dx^3} \left(\frac{\partial f}{\partial y'''} \right) = 0$$

$$\therefore 2x - \frac{d}{dx} (0) + \frac{d^2}{dx^2} (0) - \frac{d^3}{dx^3} \left(-2 \frac{d^3 y}{dx^3} \right) = 0$$

$$\therefore 2x + 2 \frac{d^6 y}{dx^6} = 0$$

$$\therefore \frac{d^6 y}{dx^6} = -x$$

On integration, $\frac{d^5 y}{dx^5} = \frac{-x^2}{2} + c_1$

Again on integration, $\frac{d^4 y}{dx^4} = \frac{-1}{2} \times \frac{x^3}{3} + c_1 x + c_2$

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Again on integration, $\frac{d^3y}{dx^3} = \frac{-1}{6} \times \frac{x^4}{4} + c_1 \times \frac{x^2}{2} + c_2x + c_3$

Again on integration, $\frac{d^2y}{dx^2} = \frac{-1}{24} \times \frac{x^5}{5} + \frac{1}{2}c_1 \times \frac{x^3}{3} + c_2 \times \frac{x^2}{2} + c_3x + c_4$

Again on integration, $\frac{dy}{dx} = \frac{-1}{120} \times \frac{x^6}{6} + \frac{1}{6}c_1 \times \frac{x^4}{4} + \frac{1}{2}c_2 \times \frac{x^3}{3} + c_3 \times \frac{x^2}{2} + c_4x + c_5$

Again on integration, $y = \frac{-1}{720} \times \frac{x^7}{7} + \frac{1}{24}c_1 \times \frac{x^5}{5} + \frac{1}{6}c_2 \times \frac{x^4}{4} + \frac{1}{2}c_3 \times \frac{x^3}{3} + c_4 \times \frac{x^2}{2} + c_5x + c_6$

Hence, the extremum of $\int_{x_0}^{x_1} (2xy - y'''')dx$ is $y = \frac{-x^7}{7!} + k_1x^5 + k_2x^4 + k_3x^3 + k_4x^2 + k_5x + k_6$

4 b) A transmission channel has a per-digit error probability of more than 1 error in 10 (06) received digits using

- i) Binominal and
- ii) Poisson distribution

Ans:

Let p = per digit error probability = 0.01; n=10

Let X denote number of digits transmitted.

Binominal Distribution;

$p = 0.01;$

$q = 1 - p = 1 - 0.01 = 0.99;$

$n = 10$

$$\therefore P(X = x) = {}^nC_x p^x q^{n-x}$$

$$\therefore P(X = x) = {}^{10}C_x (0.01)^x (0.99)^{10-x}$$

$P(\text{more than 1 error}) = P(X > 1)$

$= 1 - P(X \leq 1)$

$= 1 - [P(X=0) + P(X=1)]$

$$= 1 - [{}^{10}C_0(0.01)^0(0.99)^{10} + {}^{10}C_1(0.01)^1(0.99)^9]$$

$$= 1 - [0.9044 + 0.0913]$$

$$= 0.0043$$

Moment Generating Function of Binominal Distribution about origin is = $M_o(t) = (q + pe^t)^n$

Poisson Distribution:

$$\text{Mean}(m) = n p = 10 \times 0.01 = 0.1$$

$$\therefore P(X = x) = \frac{e^{-m} m^x}{x!}$$

$$\therefore P(X = x) = \frac{e^{-0.1} (0.1)^x}{x!}$$

$$P(\text{more than 1 error}) = P(X > 1)$$

$$= 1 - P(X \leq 1)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[\frac{e^{-0.1} \times (0.1)^0}{0!} + \frac{e^{-0.1} \times (0.1)^1}{1!} \right]$$

$$= 1 - [0.9048 + 0.0905]$$

$$= 0.0047$$

Moment Generating Function of Binominal Distribution about origin is = $M_o(t) = e^{m(e^t - 1)}$

Hence, Probability of more than 1 error by

Binominal Distribution → 0.0043 and

Poisson Distribution → 0.0047

4 c) Obtain Taylor's series and two distinct Laurent's series expansion of $f(z) = \frac{z-1}{z^2-2z-3}$ indicating the region of convergence . (08)

$$\text{Ans: } f(z) = \frac{z-1}{z^2-2z-3}$$

$$= \frac{z-1}{(z-3)(z+1)}$$

$$= \frac{\frac{1}{2}}{z+1} + \frac{\frac{1}{2}}{z-3} \rightarrow (1) \text{ (By Partial Fractions)}$$

Taylor Series:

Let $a = 0$

$$\text{From (1), } f(z) = \frac{1}{2}(z+1)^{-1} + \frac{1}{2}(z-3)^{-1}$$

$$\therefore f(a) = f(0) = \frac{1}{2}(0+1)^{-1} + \frac{1}{2}(0-3)^{-1} = \frac{1}{3}$$

$$\therefore f'(z) = \frac{-1}{2}(z+1)^{-2} - \frac{1}{2}(z-3)^{-2}$$

$$\therefore f'(a) = f'(0) = \frac{-1}{2}(0+1)^{-2} - \frac{1}{2}(0-3)^{-2} = \frac{-5}{9}$$

$$\therefore f''(z) = (z+1)^{-3} + (z-3)^{-3}$$

$$\therefore f''(a) = f''(0) = (0+1)^{-3} + (0-3)^{-3} = \frac{26}{27}$$

$$\therefore f'''(z) = -3(z+1)^{-4} - 3(z-3)^{-4}$$

$$\therefore f'''(a) = f'''(0) = -3(0+1)^{-4} - 3(0-3)^{-4} = \frac{-82}{27}$$

By Taylor Series,

$$f(z) = f(a) + (z-a)f'(a) + (z-a)^2 \frac{f''(a)}{2!} + \dots$$

$$\therefore \frac{z-1}{z^2-2z-3} = \frac{1}{3} + (z-0) \cdot \frac{-5/9}{1!} + (z-0)^2 \cdot \frac{26/27}{2!} + (z-0)^3 + \frac{-82/27}{3!} + \dots$$

$$\therefore \frac{z-1}{z^2-2z-3} = \frac{1}{3} - \frac{5}{9}z + \frac{13}{27}z^2 - \frac{41}{81}z^3 + \dots$$

Laurent's series expansion :

We consider following three cases, (Write any two Cases for Exams)

Case 1 : For $|z| < 1$,

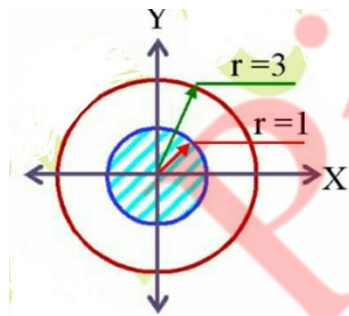
Obviously, $|z| < 3$

$$\therefore |z| < 1 \text{ and } \left| \frac{z}{3} \right| < 1$$

$$\therefore f(z) = \frac{1}{2(1+z)} + \frac{1}{2 \times 3 \left(\frac{z}{3} - 1 \right)}$$

$$= \frac{1}{2}(1+z)^{-1} - \frac{1}{6} \left(1 - \frac{z}{3} \right)^{-1}$$

$$= \frac{1}{2}(1 - z + z^2 - z^3 + \dots) - \frac{1}{6} \left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \dots \right)$$



Region of Convergence.

Above the series is convergent for $|z| < 1$ and $|z| < 3$ i.e. $|z| < 1$, which is the interior of the circle with centre (0,0) and radius 1.

Case 2: For $1 < |z| < 3$

$$\therefore 1 < |z| \text{ and } |z| < 3$$

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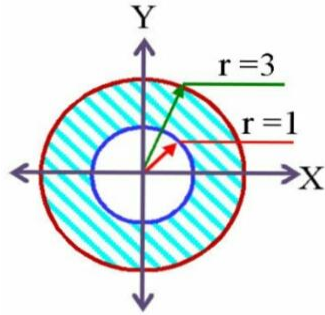
$$\therefore \left| \frac{1}{z} \right| < 1 \text{ and } \left| \frac{z}{3} \right| < 1$$

$$\therefore f(z) = \frac{1}{2z(1 + 1/z)} + \frac{1}{2 \times 3 \left(\frac{z}{3} - 1 \right)}$$

$$= \frac{1}{2z} \left(1 + \frac{1}{z} \right)^{-1} - \frac{1}{6} \left(1 - \frac{z}{3} \right)^{-1}$$

$$= \frac{1}{2z} \left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots \right) - \frac{1}{6} \left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} \dots \right)$$

$$= \frac{1}{2} \left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \dots \right) - \frac{1}{6} \left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} \dots \right)$$



Region of Convergence:

Above series is convergent for

$$\left| \frac{1}{z} \right| < 1 \text{ and } \left| \frac{z}{3} \right| < 1$$

i.e. $1 < |z| < 3$, which is the annular region between the concentric circles with centre (0,0) and radii 1 & 3.

Case 3: For $|z| > 3$,

Obviously, $|z| > 1$

$$\therefore 1 < |z| \text{ and } 3 < |z|$$

$$\therefore \left| \frac{1}{z} \right| < 1 \text{ and } \left| \frac{3}{z} \right| < 1$$

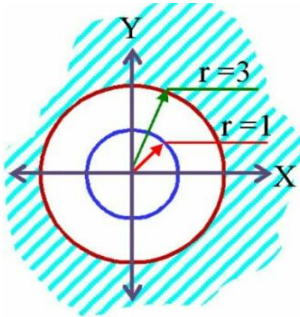
$$\therefore f(z) = \frac{1}{2z(1 + 1/z)} + \frac{1}{2z(1 - 3/z)}$$

$$= \frac{1}{2z} \left(1 + \frac{1}{z}\right)^{-1} + \frac{1}{2z} \left(1 - \frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z} \left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots\right) + \frac{1}{2z} \left(1 + \frac{3}{z} + \frac{3^2}{z^2} + \dots\right)$$

$$= \frac{1}{2} \left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \dots\right) + \frac{1}{2} \left(\frac{1}{z} + \frac{3}{z^2} + \frac{3^2}{z^3} + \dots\right)$$

Region of convergence:



Above series is convergent for $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{3}{z}\right| < 1$

i.e. $|z| > 3$, Which is the exterior region of the circle with centre (0,0) and radius 3.

5 a) verify the Cayley -Hamilton Theorem for Matrix A and hence find A^{-1} if it exists where

$$A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & z \\ b & -a & 0 \end{bmatrix}. \quad (06)$$

$$\text{Ans: } A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}.$$

Consider, $|A| = 0 - c(0 - ab) - b(ac - 0)$

$$\therefore |A| = abc - abc = 0$$

Let λ be eigen value of matrix A.

Characteristic equation is $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{vmatrix} = 0$$

On solving, we get

$$\lambda^3 - (\text{sum of diagonal elements}) \lambda^2 + (\text{sum of the minors of diagonal elements}) \lambda - |A| = 0$$

$$\therefore \lambda^3 - (0 + 0 + 0) \lambda^2 + (a^2 + b^2 + c^2) \lambda - 0 = 0$$

$$\therefore \lambda^3 + (a^2 + b^2 + c^2) \lambda = 0$$

Cayley Hamilton Theorem states that the characteristic equation is satisfied by matrix A.

$$\therefore A^3 + (a^2 + b^2 + c^2)A = 0 \rightarrow (1)$$

Now, $A^2 = A \times A$

$$= \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \times \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -c^2 - b^2 & ab & ac \\ ab & -c^2 - a^2 & bc \\ ac & bc & -b^2 - a^2 \end{bmatrix}$$

Also, $A^3 = A^2 \times A$

$$= \begin{bmatrix} -c^2 - b^2 & ab & ac \\ ab & -c^2 - a^2 & bc \\ ac & bc & -b^2 - a^2 \end{bmatrix} \times \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -abc + abc & -c^2 - b^2c - a^2c & bc^2 + b^3 + a^2b \\ c^3 + b^2c + a^2c & abc - abc & -ab^2 - ac^2 - a^3 \\ -bc^2 - b^3 - a^2b & ac^2 + ab^2 + a^3 & -abc + abc \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c(c^2 + b^2 + a^2) & b(c^2 + b^2 + a^2) \\ c(c^2 + b^2 + a^2) & 0 & -a(b^2 + c^2 + a^2) \\ -b(c^2 + b^2 + a^2) & a(c^2 + b^2 + a^2) & 0 \end{bmatrix}$$

$$= (a^2 + b^2 + c^2) \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$\therefore A^3 = -(a^2 + b^2 + c^2)A$$

$$\therefore A^3 = -(a^2 + b^2 + c^2)A = 0$$

\therefore Cayley Hamilton Theorem is verified.

Also, since $|A| = 0$, A^{-1} does not exist.

5 b) Let R^3 have the Euclidean inner product. Use Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into an orthonormal basis where

$$u_1 = (1, 1, 1), \quad u_2 = (-1, 1, 0), \quad u_3 = (1, 2, 1) \quad (06)$$

Ans: Gram Schmidt orthogonalization :

$$\text{Let } u_1 = (1, 1, 1); \quad u_2 = (-1, 1, 0); \quad u_3 = (1, 2, 1);$$

S1:

$$\text{Let } v_1 = u_1 = (1, 1, 1)$$

$$\therefore \|v_1\|^2 = (1)^2 + (1)^2 + (1)^2 = 3 \text{ and}$$

$$\langle u_2, v_1 \rangle = (-1)(1) + (1)(1) + (1)(0) = -1 + 1 + 0 = 0$$

S2:

$$\text{Let } v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$= (-1, 1, 0) - \frac{0}{3} \times (1, 1, 1)$$

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$$\therefore \|v_2\|^2 = (-1)^2 + (1)^2 + (0)^2 = 2;$$

Now,

$$\langle u_3, v_1 \rangle = (1)(1) + (2)(1) + (1)(1) = 1 + 2 + 1 = 4 \text{ and,}$$

$$\langle u_3, v_2 \rangle = (1)(-1) + (2)(1) + (1)(0) = -1 + 2 + 0 = 1$$

S3:

$$\text{Let } v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= (1, 2, 1) - \frac{4}{3}(1, 1, 1) - \frac{1}{2}(-1, 1, 0)$$

$$= (1, 2, 1) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) - \left(\frac{-1}{2}, \frac{1}{2}, 0\right)$$

$$= \left(1 - \frac{4}{3} + \frac{1}{2}, 2 - \frac{4}{3} - \frac{1}{2}, 1 - \frac{4}{3} - 0\right)$$

$$= \left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$$

$$\therefore \|v_3\|^2 = \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{-1}{3}\right)^2 = \frac{1}{6};$$

\therefore Orthonormal bases are

$$\vec{q_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{1}{\sqrt{3}}(1, 1, 1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\vec{q_2} = \frac{\vec{v_2}}{\|\vec{v_2}\|} = \frac{1}{\sqrt{2}}(-1, 1, 0) = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\text{And, } \vec{q_3} = \frac{\vec{v_3}}{\|\vec{v_3}\|} = \frac{1}{1/\sqrt{6}}\left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$$

$$= \sqrt{6}\left(\frac{1}{6}, \frac{1}{6}, \frac{-2}{6}\right) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$$

\therefore Orthonormal basis of the subspace S are

$$\left\{ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right); \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right); \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}} \right) \right\}$$

5 c) The marks obtained by 1000 students in an examination are found to be normally distributed with mean 70 and standards deviation 5. Estimate the number of students whose marks will be (08)

- i) between 60 and 75
- ii) more than 75

Ans:

Mean (m)=70

Standard deviation (σ) = 5;

N = 1000

Let X denote marks obtained by a student.

i) $P(\text{between 60 and 75 marks}) = P(60 < X < 75)$

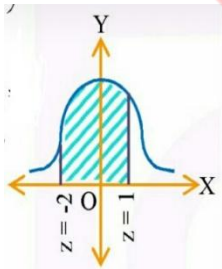
$$= P\left(\frac{60-70}{5} < \frac{X-m}{\sigma} < \frac{75-70}{5}\right)$$

$$= P(-2 < z < 1)$$

= Area between 'z=0' to 'z=1' + Area between 'z=0' to 'z=-2'

$$= 0.3413 + 0.4773$$

$$= 0.8186$$



∴ Number of students scoring between 60 and 75 marks

$$= N \times P(60 < X < 75)$$

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$$= 1000 \times 0.8186$$

$$= 819 \text{ students.}$$

$$\text{II) } P(\text{between 65 and 75 marks}) = P(65 < X < 75)$$

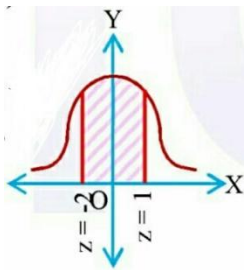
$$= P\left(\frac{65-70}{5} < \frac{X-m}{\sigma} < \frac{75-70}{5}\right)$$

$$= P(-1 < z < 1)$$

$$= \text{Area between 'z=0' to 'z=1'} + \text{Area between 'z=0' to 'z=-1'}$$

$$= 0.3413 + 0.3413$$

$$= 0.6826$$



∴ Number of students scoring between 65 and 75 marks

$$= N \times P(60 < X < 75)$$

$$= 1000 \times 0.6826$$

$$\approx 683 \text{ students.}$$

$$\text{III) } P(\text{more than 75 marks}) = P(x > 75)$$

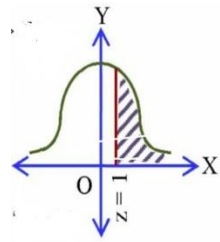
$$= P\left(\frac{X-m}{\sigma} > \frac{75-70}{5}\right)$$

$$= P(z > 1)$$

$$= 0.5 - \text{Area between 'z=0' to 'z=1'}$$

$$= 0.5 - 0.3413$$

$$= 0.1587$$



∴ Number of students scoring more than 75 marks

$$= 1000 \times 0.1587$$

$$\approx 159 \text{ students}$$

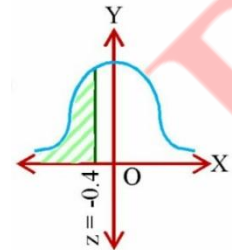
IV) P (less than 68 marks) = $P(x < 68)$

$$= P\left(\frac{X-m}{\sigma} < \frac{68-70}{5}\right)$$

$$= P(z < -0.4)$$

= 0.5 – Area between 'z=0' to z = 0.4'

$$= 0.5 - 0.1554 = 0.3446$$



∴ Number of students scoring less than 68 marks

$$= 1000 \times 0.3446$$

$$\approx 354 \text{ students}$$

Hence,

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Number of students scoring between 60 and 75 marks = 819

Number of students scoring between 65 and 75 marks = 683

Number of students scoring more than 75 marks = 159

Number of students scoring more than 75 marks = 345

6 a) Using Rayleigh- Ritz method, solve the boundary value problem $I = \int_0^1 (2xy - y^2 - y'^2) dx, 0 \leq x \leq 1$, given $y(0) = y(1) = 0$. (06)

Ans:

$$\text{Let } I = \int_0^1 (y'^2 + y^2 - 2xy) dx \rightarrow (1)$$

$$\text{Let the approximate solution be } y(x) = c_0 + c_1x + c_2x^2 \rightarrow (2)$$

$$\text{Put } x = 0, y(0) = c_0 + 0 + 0 \quad [\because y(0) = 0]$$

$$\therefore 0 = c_0 \rightarrow (3)$$

$$\text{Put } x = 1 \text{ in (2), } y(1) = c_0 + c_1 + c_2$$

$$\therefore 0 = 0 + c_1 + c_2 \quad [\text{From 3 and } y(1) = 0]$$

$$\therefore c_2 = -c_1 \rightarrow (4)$$

$$\text{Substituting (3) and (4) in (2), } y = 0 + c_1x - c_1x^2 \rightarrow (5)$$

$$\text{Differentiating W.r.t. 'x', 'y' = } c_1 - 2c_1x \rightarrow (6)$$

Substituting (5) and (6) in (1), we get

$$I = \int_0^1 [(c_1 - 2c_1x)^2 + (c_1x - c_1x^2)^2 - 2x(c_1x - c_1x^2)] dx$$

$$= \int_0^1 [(c_1^2 - 4c_1^2x + 4c_1^2x^2) + (c_1^2x^2 - 2c_1^2x^3 + c_1^2x^4) - (2c_1x^2 - 2c_1x^3)] dx$$

$$= \int_0^1 [c_1^2 - 4c_1^2x + 4c_1^2x^2 + c_1^2x^2 - 2c_1^2x^3 + c_1^2x^4 - 2c_1x^2 + 2c_1x^3] dx$$

$$= \left[c_1^2 x - \frac{4c_1^2 x^2}{2} + \frac{4c_1^2 x^3}{3} + \frac{c_1^2 x^3}{3} - \frac{2c_1^2 x^4}{4} + \frac{c_1^2 x^5}{5} - \frac{2c_1^2 x^3}{3} + \frac{2c_1 x^4}{4} \right]_0^1$$

$$= \left[c_1^2 - 2c_1^2 + \frac{4c_1^2}{3} + \frac{c_1^2}{3} - \frac{c_1^2}{2} + \frac{c_1^2}{5} - \frac{2c_1}{3} + \frac{c_1}{2} \right] - [0 - 0 + 0 + 0 - 0 + 0 - 0 + 0]$$

$$\therefore I = \frac{11}{30} c_1^2 - \frac{1}{6} c_1$$

For maximum or minimum, $\frac{dI}{dc_1} = 0$

$$\therefore \frac{11}{30} \times 2c_1 - \frac{1}{6} = 0$$

$$\therefore \frac{1}{6} = \frac{11}{15} c_1$$

$$\therefore c_1 = \frac{5}{22}$$

Hence, from (5) the approximate solution is $y = \frac{5}{22} x - \frac{5}{22} x^2$

$\therefore y = \frac{5}{22} x(x - 1)$ is the required solution.

6 b) Show that $A = \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ is derogatory and find its minimal polynomial. (06)

Ans: Let λ be eigen value of matrix A.

Characteristic equation is $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 4 - \lambda & -2 & 2 \\ 6 & -3 - \lambda & 4 \\ 3 & -2 & 3 - \lambda \end{vmatrix} = 0$$

On solving we get

$$\lambda^3 - (\text{sum of diagonal elements})\lambda^2 + (\text{sum of minors of diagonal elements})\lambda - |A| = 0$$

$$\therefore \lambda^3 - (4 - 3 + 3)\lambda^2 + (-1 + 6 + 0)\lambda - 2 = 0$$

$$\therefore \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

\therefore Eigen values (λ) are 1,1,2

$$\text{Let } f(x) = (x-1)(x-2) = x^2 - 3x + 2$$

$$\text{Now, } A^2 = A \times A = \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \times \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & -6 & 6 \\ 18 & -11 & 12 \\ 9 & -6 & 7 \end{bmatrix}$$

$$\therefore A^2 - 3A + 2I = \begin{bmatrix} 10 & -6 & 6 \\ 18 & -11 & 12 \\ 9 & -6 & 7 \end{bmatrix} - 3 \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

= 0

$\therefore f(x) = x^2 - 3x + 2$ annihilates A

$\therefore f(x)$ is a minimal polynomial.

Degree of $f(x) = 2$ & order of A=3

Since degree of $f(x) <$ order of A. Matrix A is derogatory.

6 c) Using Cauchy's residue theorem, evaluate $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$ (04)

$$\text{Ans: Let } I = \int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$$

Consider a circle $|z| = 1$ which has centre (0,0) and radius 1.

$$\text{Put } z = re^{i\theta} = le^{i\theta} = e^{i\theta}$$

$$\therefore dz = e^{i\theta} \cdot i d\theta = iz d\theta$$

$$\therefore d\theta = \frac{dz}{iz}$$

$$\text{Also, } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} = \frac{z^2 - 1}{2iz}$$

$$\text{On substituting we get, } I = \int_c \frac{1}{5+3\left[\frac{z^2-1}{2iz}\right]} \cdot \frac{dz}{iz}$$

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$$= \int_c \frac{2iz}{10iz+3(z^2-1)} \cdot \frac{dz}{iz}$$

$$= \int_c \frac{2}{3z^2+10iz-3} \cdot dz$$

For singularity, $3z^2 + 10iz - 3 = 0$

$$\therefore z = 3i \text{ or } z = \frac{-i}{3}$$

Here, ' $z_0 = -3i$ ' lies outside while $z_0 = \frac{-i}{3}$ lies inside the circle $|z|=1$

$z_0 = \frac{-i}{3}$ is a Simple pole.

R_1 = Residue of $f(z)$ at $z = \frac{-i}{3}$

$$= \lim_{z \rightarrow z_0} (z - z_0) \times f(z)$$

$$= \lim_{z \rightarrow -i/3} \cancel{(z + \frac{i}{3})} \times \frac{2}{3\cancel{(z + \frac{i}{3})}(z+3i)}$$

$$= \frac{2}{3(\frac{-i}{3}+3i)}$$

$$= \frac{1}{4i}$$

By Cauchy's Residue theorem,

$$\int_c f(z) dz = 2\pi i (R_1 + R_2 + \dots)$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = 2\pi i \times \frac{1}{4i}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = \frac{\pi}{2}$$

6 d) Using Cauchy residue theorem ,evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$, $a > 0, b > 0$. (04)

Ans:**S1:**

Consider the contour of a large semicircle with diameter on real axis, centre at origin and above the real axis.

S2:

$$\text{Let } f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

$$\text{As } z \rightarrow \infty, zf(z) \rightarrow 0$$

S3:

$$\text{For singularity, } (z^2 + a^2)(z^2 + b^2) = 0$$

$$\therefore z^2 + a^2 = 0 \text{ or } z^2 + b^2 = 0$$

$$\therefore z^2 = -a^2 \text{ or } z^2 = -b^2$$

$$\therefore z^2 = i^2 a^2 \text{ or } z^2 = i^2 b^2$$

$$\therefore z = \pm a i \text{ or } z = \pm b i$$

Here, $z_0 = -a i, -b i$ lies outside while $z_0 = a i, b i$ lies inside the contour.

$z_0 = "ai"$ and $"bi"$ are simple poles .

S4:

$$R_1 = \text{Residue of } f(z) \text{ at } "z= a i"$$

$$= \lim_{z \rightarrow z_0} (z - z_0) \times f(z)$$

$$= \lim_{z \rightarrow ai} \cancel{(z - ai)} \times \frac{z^2}{\cancel{(z - ai)}(z + ai)(z^2 + b^2)}$$

$$= \frac{(ai)^2}{(ai + ai)[(ai)^2 + b^2]}$$

$$= \frac{-a^2}{(2ai)(-a^2+b^2)}$$

$$= \frac{-a}{2i(-a^2+b^2)}$$

Similarly, interchanging “a” and “b” we get,

$R_2 = \text{Residue of } f(z) \text{ at “} z=b \text{”}$

$$= \frac{-b}{2i(-b^2+a^2)}$$

By Cauchy’s Residue theorem,

$$\int_c f(z)dz = 2\pi i(R_1 + R_2 + \dots)$$

$$\therefore \int_c \frac{z^2}{(z^2+a^2)(z^2+b^2)} dz = 2\pi i \left\{ \frac{-a}{2i(-a^2+b^2)} + \frac{-b}{2i(-b^2+a^2)} \right\}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i \cdot \frac{1}{2i} \left\{ \frac{-a}{-(a^2-b^2)} + \frac{b}{(a^2-b^2)} \right\}$$

$$= \pi \cdot \frac{1}{(a^2-b^2)} \times \{a-b\}$$

$$= \frac{\pi}{(a-b)(a+b)} \times (a-b)$$

$$= \frac{\pi}{a+b}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$$